10: Error Correcting Codes

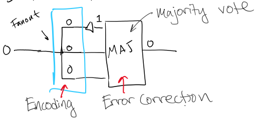
Friday, April 30, 2021 5:08 PM

Errors (why we haven't built big Q. computers)

- Classical Computers
  - Cosmic Rays
    - → 1 $\underbrace{\text{bit flip}}_{0 \leftrightarrow 1}$ per 4GB per day

Usually just don't worry about it... (although this can backfire)

Solution: Repetition Code



- Quantum computers
  - laser shape/frequency/focus bad
  - Other atom bounces into your atom
  - system is actually qudit
  - Magnetic fields
  - Electric fields
  - Heat (anomalous heating)

SO MANY ERRORS!

Why won't repetition code work for quantum?

A) MAJ not reversible

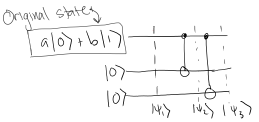
B) We can get phase errors that won't be corrected.

C) No FANOUT because can't copy quantum states.

D) All of the above.

Gil Kalai → argues quantum error correction will never work

Quantum Error Correction Encoding



$|\psi_1\rangle = a|000\rangle + b|100\rangle$

$|\psi_2\rangle = a|000\rangle + b|110\rangle$

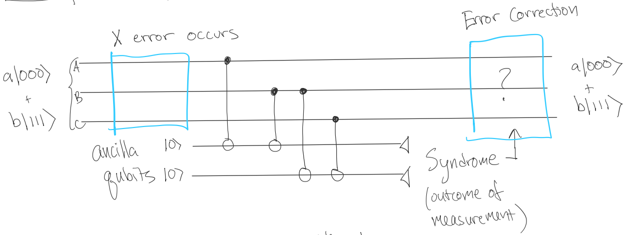
$|\psi_3\rangle = a|000\rangle + b|111\rangle$

1 "logical qubit" encoded state = 3 "physical qubits"

$a|0_L\rangle + b|1_L\rangle \longleftrightarrow \boxed{a|000\rangle + b|111\rangle}$

Similar to repetition code... but more quantum-y!

Group Work

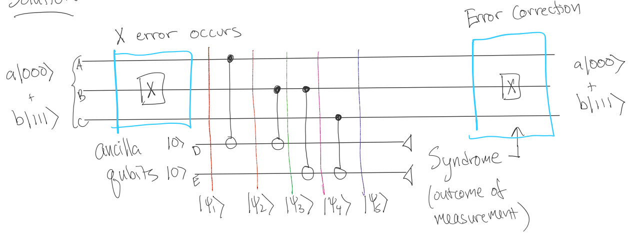


Goal: correct error without collapsing superposition $a|0\rangle_L + b|1\rangle_L$

1. Analyze outcome of ancilla measurement
   ↳ Given ancilla outcome, what state does ABC collapse to?

   Group 1, 2 : X on A
   Group 3, 4 : X on B
   Group 5, 6 : X on C
2. Based on ancilla outcome + collapsed state, what operation should be used for error correction?

Solution



$$|\psi_1\rangle = X_2\big(a|000\rangle + b|111\rangle\big)|00\rangle = a|010\rangle|00\rangle + b|101\rangle|00\rangle$$

$$|\psi_2\rangle = a|010\rangle|00\rangle + b|101\rangle|10\rangle$$

$$|\psi_3\rangle = a|010\rangle|10\rangle + b|101\rangle|10\rangle$$

$$|\psi_4\rangle = a|010\rangle|11\rangle + b|101\rangle|10\rangle$$

$$|\psi_5\rangle = a|010\rangle|11\rangle + b|101\rangle|11\rangle$$

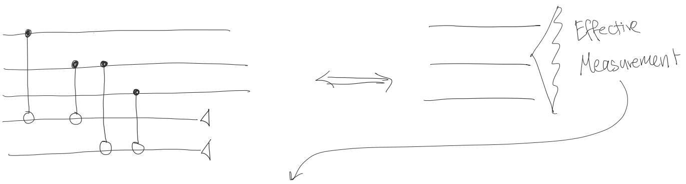
$$= \underbrace{\big(a|010\rangle + b|101\rangle\big)}_{\text{normalized}}{}_{ABC}|11\rangle_{DE} \Rightarrow$$

With probability 1: measure $|11\rangle$

ABC collapse $\Rightarrow a|010\rangle + b|101\rangle$

To correct, apply $X_2 \Rightarrow a|000\rangle + b|111\rangle$

Easier Approach to Partial Measurement



Projective Measurement:

1. Take Complete Measurement e.g. $M = \{|\phi_1\rangle, |\phi_2\rangle, \ldots |\phi_n\rangle\}$ n-dim orthonormal basis
2. Group basis states into disjoint subsets:

   e.g. $\{|\phi_1\rangle, |\phi_4\rangle\}$, $\{|\phi_3\rangle\}$, $\{|\phi_2\rangle, |\phi_5\rangle, |\phi_6\rangle\}\ldots$
3. Create Projector from each subset

$$P_0 = |\phi_1\rangle\langle\phi_1| + |\phi_4\rangle\langle\phi_4|$$

$$P_1 = |\phi_3\rangle\langle\phi_3|$$

$$P_2 = |\phi_2\rangle\langle\phi_2| + |\phi_5\rangle\langle\phi_5| + |\phi_6\rangle\langle\phi_6|$$

$$\vdots$$

$$M = \{\underset{\uparrow}{P_0}, \underset{\uparrow}{P_1}, \underset{\uparrow}{P_2}, \ldots\} \Leftarrow \text{Projective measurement description}$$

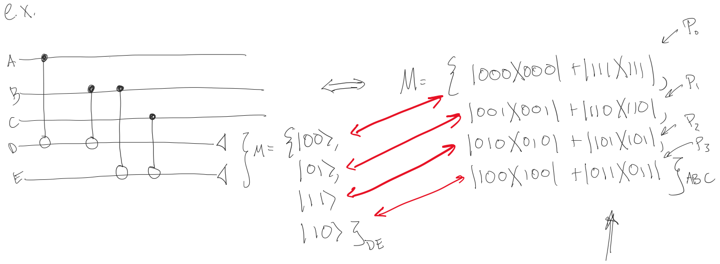
possible outcomes (fewer than complete!)

If apply M to state $|\psi\rangle$

- Probability of outcome P_i $\Rightarrow$ $\langle\psi|P_i|\psi\rangle$
- If get outcome P_i, $|\psi\rangle$ collapses to $\underbrace{P_i|\psi\rangle \text{ normalized}}$

$$\frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

e.x.



$$\Longleftrightarrow \quad M = \{\, |000\rangle\langle 000| + |111\rangle\langle 111|, \; |001\rangle\langle 001| + |110\rangle\langle 110|, \; |010\rangle\langle 010| + |101\rangle\langle 101|, \; |100\rangle\langle 100| + |011\rangle\langle 011| \,\}_{ABC}$$

P_0 P_1 P_2 P_3

$$M = \{ |00\rangle, |01\rangle, |11\rangle, |10\rangle \}_{DE}$$

Q: If measure $\frac{1}{\sqrt{6}}|000\rangle_{ABC} + \sqrt{\frac{2}{6}}|001\rangle_{ABC} + \sqrt{\frac{3}{6}}|110\rangle_{ABC}$ with M, which outcomes are possible?

A) P_0

B) P_1

C) P_2

D) P_3

Q: If measure $\frac{1}{\sqrt{6}}|000\rangle_{ABC} + \sqrt{\frac{2}{6}}|001\rangle_{ABC} + \sqrt{\frac{3}{6}}|110\rangle_{ABC}$ with M, and get outcome $P_1 = |001\rangle\langle 001| + |110\rangle\langle 110|$, what is the resultant state?

$P_1|\psi\rangle = \sqrt{\frac{2}{6}}|001\rangle + \sqrt{\frac{3}{6}}|110\rangle$ (not normalized!)

$\langle\psi|P_1|\psi\rangle = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$

$\Rightarrow \sqrt{\frac{2}{5}}|001\rangle + \sqrt{\frac{3}{5}}|110\rangle$

Big Idea with Error Correction

Partial collapse just tells you enough to correct error, doesn't collapse/destroy quantum state

But… continuous errors!

Suppose $|\psi\rangle$ is encoded

Error 1: $|\psi\rangle \to |\psi_1\rangle$, detected by P_1

Error 2: $|\psi\rangle \to |\psi_2\rangle$, detected by P_2

outcomes of projective measurement

$P_1|\psi_1\rangle = |\psi_1\rangle$

$P_2|\psi_2\rangle = |\psi_2\rangle$

What if get combined unitary:

$$|\psi\rangle \rightarrow \alpha|\psi_1\rangle + \beta|\psi_2\rangle$$

need to be able to correct for every possible α, β?!

Measure: • outcome P_1

$$P_1(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \xrightarrow[\text{to}]{\text{collapses}} |\psi_1\rangle$$

• outcome P_2

$$P_2(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) \xrightarrow[\text{to}]{\text{collapses}} |\psi_2\rangle$$

Collapse gets rid of partial rotation errors! Only need to be able to correct "big" errors, like X, Z, etc.

(See Pset)

Code we've seen can't correct Z errors. (See Pset)

Correct More Errors? Bigger Code!

$$|0_L\rangle = (|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$$
$$|1_L\rangle = (|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

} 9-qubit Shor code

$a|0\rangle + b|1\rangle$ gets encoded as $a|0_L\rangle + b|1_L\rangle$

(Any qubit rotation can be expressed as a combo of $\hat{X}, \hat{Z}$ rotations... corrects any single qubit error.

Tutoring, Reflection

Projector Formed from States: $(X_1 = X$ acting on first qubit$)$

$$P_0 \Leftarrow \{|0_L\rangle, |1_L\rangle\} = \{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle), (|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\}$$

$$P_{X1} \Leftarrow \{X_1|0_L\rangle, X_1|1_L\rangle\} = \{(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle), (|100\rangle-|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\}$$

$$P_{Z1} \Leftarrow \{Z_1|0_L\rangle, Z_1|1_L\rangle\} = \{(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle), (|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\}$$

$$P_{Y1} \Leftarrow \{Y_1|0_L\rangle, Y_1|1_L\rangle\} = \{(|100\rangle-|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle), (|100\rangle+|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\}$$

P_{X2}

⋮

Q: How do you tell if Z_1 or Z_2 occured?

A) Project onto P_{Z1} or P_{Z2}

B) Can't distinguish, so code fails

C) Can't distinguish, but doesn't matte

Z_1

$$(|000\rangle \overset{-}{\underset{-}{+}} |111\rangle)(|000 + |111\rangle)(|000\rangle+|111\rangle)$$
$$(|000\rangle \overset{+}{\underset{+}{-}} |111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$$

Z_2

Can correct either by applying Z to one of the first 3 qubits.

Q: Which errors can be corrected by Shor's code?
(Corrects by assuming fewest number of errors possible occured)

✓ A) $Z_1 Z_2$

✗ B) $Z_1 Z_4$

✓ C) $Z_1 Z_2 Z_3$

✓ D) $X_1 Z_2$

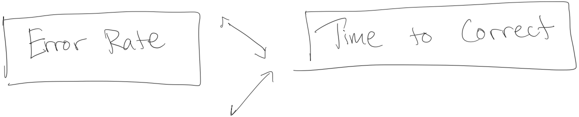
E) $X_1 X_2$ ✗

F) $X_1 X_4$ ✓

G) $Y_1 Z_4$ ✗

Have Codes... Will Correct?

Shor code → corrects all 1-qubit error, not all 2-qubit errors

Need time to correct before 2nd error arrives